1. Prove that the sum of three consecutive integers is divisible by 3.

Three consecutive integers: \[ n \quad n+1 \quad n+2 \]

Sum of:
\[ n + (n+1) + (n+2) \]
\[ = 3n + 3 \]
\[ = 3(n+1) \]
\[ \therefore \text{divisible by 3.} \]

2. Prove \((n + 6)^2 - (n + 2)^2\) is always a multiple of 8

\[
\begin{align*}
(n+6)(n+6) & \quad \quad n^2 + 12n + 36 \\
(n+2)(n+2) & \quad \quad n^2 + 4n + 4 \\
= n^2 + 12n + 36 - n^2 - 4n - 4 & \quad \quad = 8n + 32 \\
= 8(n+4) & \quad \quad \therefore \text{divisible by 8 and is therefore a multiple of 8}
\end{align*}
\]
3. Prove \((n + 10)^2 - (n + 5)^2\) is always a multiple of 5

\[
\begin{align*}
(n+10)(n+10) & - (n+5)(n+5) \\
= n^2 + 20n + 100 & - (n^2 + 10n + 25) \\
= n^2 + 20n + 100 & - n^2 - 10n - 25 \\
= 10n + 75 & \\
= 5(2n + 15) & \\
\therefore \text{ a multiple of 5} & \\
\end{align*}
\]

4. Prove the sum of two consecutive odd numbers is even.

Two consecutive odd numbers: \(2n+1\), \(2n+3\)

Sum of:

\[
\begin{align*}
= 2n+1 + 2n+3 \\
= 4n+4 \\
= 4(n+1) & \\
\therefore \text{ an even number.} & \\
\end{align*}
\]
5. \[(2n + 1)(3n - 2) - (6n - 1)(n - 2)\] is always even

\[
6n^2 + 3n - 4n - 2 - (6n^2 - n - 12n + 2)
\]

\[
= 6n^2 - n - 2 - 6n^2 + 13n - 2
\]

\[
= 12n - 4
\]

\[
= 4(3n - 1)
\]

\[
\therefore \text{ an even outcome.}
\]
7. Prove the sum of four consecutive odd numbers is always a multiple of 8

four consecutive odd numbers: \(2n+1\)
\(2n+3\)
\(2n+5\)
\(2n+7\)

Sum of:
\[
(2n+1 + 2n+3 + 2n+5 + 2n+7)
= 8n + 16
= 8(n+2)
\]

\(\therefore\) a multiple of 8.

(4)

8. Prove \((2n + 9)^2 - (2n + 5)^2\) is always a multiple of 4

\[
(2n+9)(2n+9) - (2n+5)(2n+5)
= 4n^2 + 36n + 81 - (4n^2 + 20n + 25)
= 4n^2 + 36n + 81 - 4n^2 - 20n - 25
= 16n + 56
= 4(4n + 14)
\]

\(\therefore\) a multiple of 4

(4)
9. Prove \((n + 1)^2 + (n + 3)^2 - (n + 5)^2 = (n + 3)(n - 5)\)

\[
(n+1)(n+1) + (n+3)(n+3) - (n+5)(n+5)
= n^2 + 2n + 1 + (n^2 + 6n + 9) - (n^2 + 10n + 25)
= n^2 + 2n + 1 + n^2 + 6n + 9 - n^2 - 10n - 25
= n^2 - 2n - 15
= (n + 3)(n - 5)
\] (4)

10. Prove the product of two even numbers is always even

Even numbers: \(2n\), \(2m\)

Product: \(2n \times 2m = 4mn\)

\(\Rightarrow 2(2mn)\)

\(\Rightarrow\) always even. (3)
11. Prove the product of three consecutive odd numbers is odd

Three consecutive odd numbers: \(2n+1\)
\(2n+3\)
\(2n+5\)

Product: \((2n+1)(2n+3)(2n+5)\)
\[8n^3 + 36n^2 + 46n + 15\]
\[2\left(4n^3 + 18n^2 + 23n\right) + 15\]
\[
\text{Even} \quad + \quad \text{odd} = \text{odd}
\]

(3)

12. Prove algebraically that the sum of the squares of two odd integers is always even.

\(2n + 1\) and \(2k + 1\)

\((2n + 1)^2 + (2k + 1)^2\)
\[= 4n^2 + 4n + 1 + 4k^2 + 4k + 1\]
\[= 4n^2 + 4n + 4k^2 + 4k + 2\]
\[= 2(2n^2 + 2n + 2k^2 + 2k + 1)\]
\[\therefore \text{always even}\]

(4)
13. Prove that when two consecutive integers are squared, that the difference is equal to the sum of the two consecutive integers.

Two consecutive integers: \( n \), \( n+1 \)

Sum of: \( n + (n+1) \)
\[ = 2n+1 \]

Difference of squares:
\[ = (n+1)^2 - n^2 \]
\[ = (n+1)(n+1) - n^2 \]
\[ = n^2 + 2n + 1 - n^2 \]
\[ = 2n + 1 \]

\[ \therefore \text{equal to the sum of.} \quad (4) \]

14. Prove algebraically that

\((4n + 1)^2 - (2n - 1)\) is an even number

for all positive integer values of \( n \).

\[ (4n+1)(4n+1) - (2n-1) \]
\[ = 16n^2 + 8n + 1 - 2n + 1 \]
\[ = 16n^2 + 6n + 2 \]
\[ = 2(8n^2 + 3n + 1) \]
\[ \therefore \text{even outcome} \quad (4) \]
15. Prove that \(3n(3n + 4) + (n - 6)^2\) is positive for all values of \(x\).

\[
9n^2 + 12n + (n^2 - 6n - 6n + 36)
\]
\[
10n^2 + 36
\]

as \(n \geq 0\)

\(n^2 > 0\)

\(10n^2 > 0\)

\(10n^2 + 36 > 0\)

\[\therefore \quad 10n^2 + 36 \text{ is always positive.}\]

16. The first five terms of a linear sequence are 5, 11, 17, 23, 29 ...

(a) Find the \(n\)th term of the sequence

\[6n - 1\]

A new sequence is generated by squaring each term of the linear sequence and then adding 5.

(b) Prove that all terms in the new sequence are divisible by 6.

\[
(6n - 1)^2 + 5
\]
\[
(36n^2 - 12n + 6n + 1) + 5
\]
\[
36n^2 - 12n + 6
\]
\[
6(6n - 2n + 1)
\]

\[\therefore \text{ divisible by 6.}\]
17. Prove that the product of two consecutive even numbers is a multiple of 4.

Two consecutive even numbers: \(2n\) and \(2n + 2\)

Product: \(2n(2n+2)\)

\[4n^2 + 4n\]

\[4n(n+1)\]

\[\therefore \text{ a multiple of } 4\]  

(3)

18. Prove that when any odd integer is squared, the result is always one more than a multiple of 8.

Odd integer: \(2n + 1\)

Odd integer squared: \((2n+1)(2n+1)\)

\[= 4n^2 + 4n + 1\]

\[= 4n(n+1) + 1\]

Since either \(n\) or \(n+1\) is even, \(n(n+1)\) is even.

4 times even is always a multiple of 8.

\[\therefore 4n(n+1)\text{ is a multiple of } 8\]

\[\therefore 4n(n+1)+1\text{ is one more than a multiple of } 8\]  

(4)

19. Prove that the product of two odd numbers is always odd.

Odd numbers: \(2k+1\) and \(2m+1\)

Product: \((2k+1)(2m+1)\)

\[4km + 2k + 2m + 1\]

\[2(2km + k + m) + 1\]

Even + odd = odd.  

(3)
20. Given $2^{89} - 1$ is prime.

Show that $2^{89} + 1$ is a multiple of 3

Let $n, n+1, n+2$ be 3 consecutive numbers.

One number must be a multiple of 3.

$2^{89} - 1$ is prime \(\therefore\) not a multiple of 3.

$2^{89}$ is not a multiple of 3 (considering prime factors).

$2^{89} + 1$ \(\therefore\) must be a multiple of 3.

(3)